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Correlation Functions of Odd Numbers of Spins with Finite Separations on the Onsager–Ising Lattice

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A method is developed for making exact evaluations of correlation functions of odd numbers of spins on the Onsager–Ising lattice, applicable to cases in which the separations between the spins are finite. The method is based on an identity which permits the reduction of determinants of infinite-dimensional matrices to those of finite dimension. Particularly simple results are obtained when all spins are on a straight line. Numerical calculations are carried out for a few cases.

KEY WORDS: Statistical mechanics; Onsager–Ising lattice; spin; correlation function; Pfaffian; determinants of infinite-dimensional matrices; Fisher lattice; Kasteleyn lattice.

1. INTRODUCTION

Recently various authors have examined correlation functions of spins on the Onsager–Ising lattice in the limit of infinite spin separations.[1,2] The investigations have been motivated, in part, by the importance of these functions in scaling and renormalization group theory and by the close connection to ϕ^4 theory.[3]

In contrast, relatively little work has been done on correlation functions for small spin separations, possibly because of lack of comparable incentive. Further, for an odd number of spins with small separation, calculational difficulties may have inhibited investigations; only two cases[4,5] have been evaluated, each involving three spins on the same straight line.

In a recent work[6] by one of us on interacting Frenkel excitons, the moments of optical transition spectra were related to spin correlation functions. The most important correlations in this problem are those involving near neighbors. The initial motivation for the present work was the need to calculate the function $\langle\sigma_{00}\sigma_{01}\sigma_{10}\rangle$. An identity developed below

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proved to be useful for evaluating all correlation functions of a small, odd number of spins separated by small distances. The identity leads to a particularly simple expression for the correlation function if the spins lie on the same horizontal, vertical, or diagonal line.

The construction due to Montroll *et al.*[7] (MPW) for calculating the correlation function of a pair of spins may be applied in evaluating the correlation function of any even number of spins.[8] Below the Curie temperature, the correlation of an odd number is equal to $\mathcal{M}^{-1}$ times the correlation function of an even number, taken in the limit in which one spin is moved infinitely far[4,5,8] [2] from the others. Here, $\mathcal{M}$ is the spontaneous magnetization[9,10]

$$\mathcal{M}^2 = \lim_{n\to\infty} \langle \sigma_{00}\sigma_{0n} \rangle = [1 - (\sinh 2\beta E_1 \sinh 2\beta E_2)^{-2}]^{1/4}$$

where E_1 and E_2 are the horizontal and vertical Onsager–Ising coupling constants.

In applying the work of MPW to the many-spin case, one draws a series of nonintersecting lines (deformation lines) on the Onsager–Ising lattice, each line connecting a pair of spins contained within the set of those whose correlation is desired. A deformation of an intersite dimer on the Fisher[11] (or Kasteleyn[12]) lattice is associated with each segment connecting nearest neighbor spins of a given deformation line, an index (deformed site index) is assigned to each point where a deformed dimer can terminate or commence, and a mesh is defined by the direct product of the set of deformed site indices (ordered in a particular way) with itself. On this mesh, a matrix is constructed whose Pfaffian evaluates the many-spin correlation function.

When the correlation function is that of an odd number of spins, the construction leads to an infinite-dimensional determinant of a particular structure. In the next section, a theorem will be proved which permits the reduction to a deteminant of finite dimensions. In the following sections, the theorem is applied to the calculation of correlation functions of arbitrary odd sets of spins, with particular applications to $\langle \sigma_{00}\sigma_{01}\sigma_{10} \rangle$ and to cases in which the spins are on the same line.

2. REDUCTION OF INFINITE DETERMINANTS

It will be shown that for any finite-dimensional matrix of the form

$$\lambda = \begin{pmatrix} a & B \\ C & T \end{pmatrix} \tag{1}$$

[2] No one has shown that when correlation functions of an odd number of spins are calculated in this way, the result is independent of the path over which the last spin is moved to infinity.

with T invertible and a square,

$$\det \lambda = \det T \det(a - BT^{-1}C) \tag{2}$$

We first prove Eq. (2) when a is invertible. It can be seen that

$$\begin{aligned}(\det \lambda)^2 &= \det\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\det\begin{pmatrix}a & B\\ C & T\end{pmatrix}\det\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\det \lambda\\ &= (\det a \det T)^2 \det\begin{pmatrix}1 & -a^{-1}B\\ -T^{-1}C & 1\end{pmatrix}\det\begin{pmatrix}1 & a^{-1}B\\ T^{-1}C & 1\end{pmatrix}\\ &= (\det a \det T)^2 \det(1 - a^{-1}BT^{-1}C)\det(1 - T^{-1}Ca^{-1}B)\end{aligned} \tag{3}$$

The last step follows from the result of the matrix multiplication. We now show that the last two factors of the right side of Eq. (3) are equal. Let $X \equiv T^{-1}C$ and $Y \equiv a^{-1}B$, and compare $\det(1 - XY)$ with $\det(1 - YX)$. The dimensions of X are $n \times m$, with $n > m$, while those of Y are $m \times n$. Define the $n \times n$ matrix X' by adding $n - m$ columns of zeros to X, all to the right of the X columns; similarly, define Y' by adding $n - m$ rows of zeros below the rows of Y. Examination shows that $X'Y' = XY$ [so that $\det(1 - X'Y') = \det(1 - XY)$], while

$$1 - Y'X' = \begin{pmatrix}1 - YX & 0\\ 0 & 1\end{pmatrix} \tag{4}$$

Then $\det(1 - Y'X') = \det(1 - YX)$, so the problem is reduced to showing that $\det(1 - X'Y') = \det(1 - Y'X')$. Now the square matrices $X'Y'$ and $Y'X'$ have the same eigenvalue spectrum and therefore so do $(1 - X'Y')$ and $(1 - Y'X')$. Then the determinants are equal, and so

$$[\det \lambda]^2 = [\det T \det(a - BT^{-1}C)]^2 \tag{5}$$

By continuity, starting from those λ matrices in which B and C vanish, one finds that the appropriate sign choice leads to Eq. (2). Since the rhs of Eq. (2) does not involve a^{-1}, a continuity argument can be used to show it is valid when a is not invertible. The theorem can also be extended to T of countably infinite dimension if (a) $\det T$ exists and (b) the index sums in $BT^{-1}C$ converge uniformly. The theorem is useful for calculations when the division of λ into blocks fulfills the following requirements: (c) the square block a in the upper left corner is reasonably small, (d) the determinant of block T can be evaluated in the limit of infinite dimension, and (e) the matrix product $BT^{-1}C$ can be evaluated.

3. THE λ MATRICES

We will evaluate the correlation function of three spins, two of them nearest neighbors along the horizontal axis, the third a nearest neighbor of one of the two but located in the adjacent row. We will use the identity

$$|\langle\sigma_{00}\sigma_{01}\sigma_{10}\rangle| = |\mathcal{M}|^{-1} \lim_{n\to\infty} |\langle\sigma_{00}\sigma_{01}\sigma_{10}\sigma_{1n}\rangle| \tag{6}$$

The deformation geometry consists of a horizontal line from (1, 0) to (1, n) and the single line segment between (0, 0) and (0, 1) on the Onsager–Ising lattice. The set of deformed site indices on the Fisher or Kasteleyn lattice corresponding to this choice is taken in the order

$$00\text{R},\ 01\text{L},\ 10\text{R},\ 11\text{R},\ldots,\ 1(n-1)\text{R},\ 11\text{L},\ 12\text{L},\ldots,\ 1n\text{L}$$

MPW's construction leads to

$$\lim_{n\to\infty} \langle\sigma_{00}\sigma_{01}\sigma_{10}\sigma_{1n}\rangle^2 = \lim_{n\to\infty} \det[(y^{-1} + Q)(1 - z_1{}^2)] \tag{7}$$

where

$$z_i = \tanh \beta E_i, \qquad i = 1, 2 \tag{8}$$

$$y^{-1} = \begin{pmatrix} H & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} (z_1^{-1} - z_1)^{-1} \tag{9}$$

$$Q = \begin{pmatrix} V' & D' & G' \\ \tilde{D}' & 0 & -\tilde{V}'' \\ -\tilde{G}' & V'' & 0 \end{pmatrix} \tag{10}$$

The tilde designates the transpose. Corresponding blocks of y^{-1} and Q have the same dimensions; going down the diagonal of either matrix, the square diagonal blocks have dimensions 2, n, and n. The submatrices of y^{-1} and Q are

$$H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$V' = \begin{bmatrix} 0 & A^{-1}(0, 0; 0, 1)_{\text{RL}} \\ A^{-1}(0, 1; 0, 0)_{\text{LR}} & 0 \end{bmatrix} \tag{11a}$$

$$V''_{ij} = A^{-1}(1, 1+i; 1, j)_{\text{LR}} = A^{-1}(0, 1+i-j; 0, 0)_{\text{LR}}$$
$$= -A^{-1}(0, 0; 0, 1+i-j)_{\text{RL}} \tag{11b}$$

$$G'_{tj} = \delta_{t0} A^{-1}(0, 0; 1, j+1)_{\mathrm{RL}} + \delta_{t1} A^{-1}(0, 1; 1, j+1)_{\mathrm{LL}} \tag{11c}$$

$$D'_{tj} = \delta_{t0} A^{-1}(0, 0; 1, j)_{\mathrm{RR}} + \delta_{t1} A^{-1}(0, 1; 1, j)_{\mathrm{LR}} \tag{11d}$$

where

$$\begin{gathered} A^{-1}(i, j; k, l)_{\alpha\beta} = (2\pi i)^{-2} \oint\oint dv\, d\xi\, \xi^{i-k-1} v^{j-l-1} A^{-1}(v, \xi)_{\alpha\beta} \\ \alpha = \mathrm{R, L, U, D}; \qquad \beta = \mathrm{R, L, U, D} \end{gathered} \tag{11e}$$

and $A^{-1}(v, \xi)$ is found in Eq. (26) of MPW, with $\xi = e^{i\phi_1}$ and $v = e^{i\phi_2}$. The contour integrations are carried out on unit circles.

In our calculations, we require only the RL, LR, and RR elements of $A^{-1}(v, \xi)$:

$$A^{-1}(v, \xi)_{\mathrm{RR}} = -A^{-1}(v, \xi)_{\mathrm{LL}} = -(\xi^2 - 1)[\Delta(v, \xi)]^{-1} \tag{12a}$$

$$\begin{aligned} A^{-1}(v, \xi)_{\mathrm{RL}} &= -A^{-1}(v^{-1}, \xi^{-1})_{\mathrm{LR}} \\ &= -\xi[\Delta(v, \xi)]^{-1}[z_2^{-1} - z_2 - z_1 v(z_2^{-1} + \xi^{-1})(1 + z_2\xi)] \end{aligned} \tag{12b}$$

where[3]

$$\Delta(v, \xi) = (1 - z_1{}^2)[\xi - \alpha(v)][\xi - \alpha(v)^{-1}] \tag{13a}$$

$$\begin{aligned} [\alpha(v)]^{\pm 1} = {} & \{[1 - (\alpha_1\alpha_2)^{-1}](\alpha_1 - \alpha_2)\}^{-1} \\ & \times \{[f_1(v) f_1(v^{-1})]^2 \alpha_1^{-1} + [f_2(v) f_2(v^{-1})]^2 \alpha_2^{-1} \\ & \pm 2(\alpha_1\alpha_2)^{-1/2} f_1(v) f_1(v^{-1}) f_2(v) f_2(v^{-1})\} \end{aligned} \tag{13b}$$

$$\alpha_1 = z_1(1 - |z_2|)(1 + |z_2|)^{-1} \tag{13c}$$

$$\alpha_2 = z_1^{-2}\alpha_1 \tag{13d}$$

and

$$f_j(v) = (1 - \alpha_j v)^{1/2}; \qquad j = 1, 2 \tag{13e}$$

MPW found that

$$A^{-1}(0, 1 + i - j; 0, 0)_{\mathrm{LR}} = (1 - z_1{}^2)^{-1}[a_{i-j} - z_1\delta_{ij}] \tag{14a}$$

where

$$a_{i-j} = (2\pi i)^{-1} \oint \phi(v) v^{j-i-1}\, dv \tag{14b}$$

$$\phi(v) = f_1(v) f_2(v^{-1})[f_2(v) f_1(v^{-1})]^{-1} \tag{14c}$$

[3] The sign of the right side of Eq. (13b) changes when E_1/E_2 is negative; however, in this paper we use only $|\alpha(v)|^{-1}$.

From a similar calculation, we find the elements of D' and G',

$$G'_{1,m} = -D'_{0,m} = -(2\pi i)^2 \oint\oint A^{-1}(v,\xi)_{\mathrm{RR}} v^{m-1}\, dv\, d\xi$$

$$= -[2\pi i(1 - z_1{}^2)]^{-1} \oint \alpha(v)^{-1} v^{m-1}\, dv \tag{15a}$$

$$D'_{1,m} = (2\pi i)^{-2} \oint\oint A^{-1}(v,\xi)_{\mathrm{LR}} v^{-m}\xi^{-2}\, dv\, d\xi$$

$$= [(2\pi i)(1 - z_1{}^2)]^{-1} \oint \phi(v)[\alpha(v)]^{-1} v^{m-1}\, dv \tag{15b}$$

$$G'_{0,m} = -[2\pi i(1 - z_1{}^2)]^{-1} \oint v^{m-1}\phi(v^{-1})[\alpha(v)]^{-1}\, dv \tag{15c}$$

We put the matrix $\lambda' \equiv (y^{-1} + Q)(1 - z_1{}^2)$ [which appears in Eq. (7)] in the form of λ given in Eq. (1), with a 2×2 antisymmetric submatrix a whose lower left element is a_0; the submatrix T is $(2n \times 2n)$-dimensional of the form

$$T = \begin{pmatrix} 0 & -\tilde{S} \\ S & 0 \end{pmatrix} \tag{16}$$

where $S_{ij} = a_{i-j}$. The submatrices B and C are given by

$$B = (D \quad G) = -\tilde{C} \tag{17}$$

where $D = (1 - z_1{}^2)D'$ and $G = (1 - z_1{}^2)G'$.

In the more general problem (the evaluation of the $2n + 1$ spin correlation function in which all spins are separated by finite distances) we also work with a geometry in which one of the spins is connected by a horizontal or vertical deformation line to a spin infinitely far from the others. The mesh is constructed from an ordering in which indices on the line of infinite length appear last, with all R indices in ascending order, followed by all L indices in ascending order. Then, the square of the correlation function will equal the determinant of a matrix λ'', which, when arranged in the form of Eq. (1), has the T of Eq. (16); the determinant of the submatrix a equals the square of the correlation function of the $2n$ spins which are paired by the n non-intersecting lines of finite length; B and C are put in the form of Eq. (17). If the subset of deformed indices corresponding to lines of finite length has as its tth element $(i'k'\alpha)$ and if the spin which is paired with the spin at infinity is located on the Onsager–Ising lattice at (i, k), then the tjth element of D is given by

$$D_{tj} = (2\pi i)^{-1} \oint D_t(\beta)\beta^{j-1}\, d\beta \tag{18}$$

where

$$D_t(\beta) = (2\pi i)^{-1} \oint A_{\alpha R}^{-1}(\beta, \xi)\xi^{-(i'-i+1)}\beta^{(k-k')}\, d\xi \tag{19}$$

When the index R on the rhs of Eq. (19) is changed to L, the left sides of Eqs. (18) and (19) are replaced by G_{tj} and $G_t(\beta)$, respectively.

Since all RR and LL elements of $A^{-1}(0i; 0j)$ vanish, an alternate ordering simplifies the form of the matrix when all spins are in a horizontal line: all R indices precede all L indices; within each subset, the indices of all points on all deformation segments are taken in their order on the line, proceeding toward the spin at infinity. By interchanging E_1 and E_2, the correlation for spins located on the vertical axis is derived. One may use Stephenson's method[13] (also Ref. 14, pp. 186–189) to rotate spins along the horizontal to the diagonal by setting α_1 equal to zero and α_2 equal to $(\sinh 2\beta E_1 \sinh 2\beta E_2)^{-1}$ at the end of the calculation. The alternate ordering leads to

$$\left[\lim_{n\to\infty} \langle \sigma_{0,i_0}\sigma_{0,i_1} \cdots \sigma_{0,i_{2J}}\sigma_{0,i_{2J+n}}\rangle\right]^2$$

$$= \lim_{n\to\infty} \det\begin{pmatrix} 0 & -\lambda''' \\ \lambda''' & 0 \end{pmatrix} = \lim(\det \lambda''')^2 \tag{20a}$$

where

$$\lambda''' = \begin{pmatrix} \gamma & B \\ C & S \end{pmatrix} \tag{20b}$$

The square matrix S is defined as before; γ is divided into submatrices $\gamma_{2p,2q}$:

$$\gamma = \begin{pmatrix} \gamma_{00} & \gamma_{02} & \cdots & \gamma_{0,2J-2} \\ \gamma_{20} & \gamma_{22} & \cdots & \gamma_{2,2J-2} \\ \vdots & \vdots & & \vdots \\ \gamma_{2J-2,0} & \gamma_{2J-2,2} & \cdots & \gamma_{2J-2,2J-2} \end{pmatrix} \tag{20c}$$

Here, each block $\gamma_{2p,2q}$ has horizontal dimension $i_{2q+1} - i_{2q}$ and vertical dimension $i_{2p+1} - i_{2p}$, and its cl element is

$$(\gamma_{2p,2q})_{cl} = a_{i_{2p}-i_{2q}} + c - l \tag{20d}$$

with a_j defined in Eqs. (14b) and (14c);

$$\tilde{B} = (\tilde{B}_0 \quad \tilde{B}_2 \quad \cdots \quad \tilde{B}_{2J-2}) \tag{21a}$$

where each $B_{2\alpha}$ has vertical dimension $i_{2\alpha+1} - i_{2\alpha}$;

$$C = (C_0 \quad C_2 \quad \cdots \quad C_{2J-2}) \tag{21b}$$

with each $C_{2\alpha}$ having horizontal dimension $i_{2\alpha+1} - i_{2\alpha}$. The elements of $B_{2\alpha}$ and $C_{2\alpha}$ are given by

$$(B_{2\alpha})_{p,q} = a_{i_{2\alpha} - i_{2J} + p - q} \tag{21c}$$

$$(C_{2\beta})_{k,m} = a_{i_{2J} - i_{2\beta} + m - k} \tag{21d}$$

4. EVALUATION OF DETERMINANTS

The λ matrices in the last section are put in the form of Eq. (1) in such a way that conditions (a)–(e) are satisfied. The fulfillment of (a) and (d) follows from the MPW evaluation (which leads to det $S = M^2$). That (b) and (e) are satisfied in the cases treated here follows from the development below.

The product $BT^{-1}C$ contains contributions of the forms $XS^{-1}\tilde{Y}$ and $X\tilde{S}^{-1}\tilde{Y}$, where X and Y are submatrices (for example, D and G) with elements of the form

$$X_{ij} = (2\pi i)^{-1} \oint v^{j-1} X_i(v)\, dv \tag{22a}$$

$$\tilde{Y}_{km} = (2\pi i)^{-1} \oint (\beta)^{k-1} Y_m(\beta)\, d\beta \tag{22b}$$

$X_i(v)$ and $Y_k(\beta)$ are each analytic in the neighborhood of the appropriate unit circle, for all $T < T_c$. A matrix element of S^{-1} is (Ref. 14, pp. 203–215)

$$S^{-1}_{m,n} = (2\pi i)^{-2} \oint\!\!\oint_{|\lambda\xi|<1} [(1 - \lambda\xi)\lambda\xi]^{-1}\xi^{-m}\lambda^{-n}[P(\xi)]^{-1}P(\lambda)\, d\lambda\, d\xi \tag{23a}$$

$$P(\xi) = f_1(\xi)[f_2(\xi)]^{-1} \tag{23b}$$

The double contour in Eq. (23a) is taken with $|\lambda| = 1 = |\xi|$ except for a deformation $|\lambda\xi| < 1$ near $\lambda = \xi$.

In each matrix element of $XS^{-1}\tilde{Y}$, the index sums may be brought inside the quadruple integral if contours are chosen with $v < \xi$, $\beta < \lambda$. With these choices, one may sum the two geometric series, and the resulting poles at

$\nu = \xi$ and $\beta = \lambda$ can be used to perform the ξ and λ integrations, leading to

$$(XS^{-1}\tilde{Y})_{l,m} = (2\pi i)^{-2} \oint\oint_{|\nu\beta|<1} d\nu\, d\beta\, [(1-\nu\beta)\nu\beta P(\nu)]^{-1} P(\beta) X_l(\nu) Y_m(\beta) \tag{24a}$$

$$(X\tilde{S}^{-1}\tilde{Y})_{l,m} = (2\pi i)^{-2} \oint\oint_{|\nu\beta|<1} d\nu\, d\beta\, [(1-\nu\beta)\nu\beta P(\nu)]^{-1} P(\beta) X_l(\beta) Y_m(\nu) \tag{24b}$$

Equation (2) yields

$$\det \lambda'' = \mathcal{M}^4 \det(a + DS^{-1}\tilde{G} - G\tilde{S}^{-1}\tilde{D}) \tag{25}$$

$$\det \lambda''' = \mathcal{M}^2 \det(\gamma - BS^{-1}C) \tag{26}$$

We may evaluate det λ' as a special case of Eq. (25). It has been noted that

$$a = \begin{pmatrix} 0 & -a_0 \\ a_0 & 0 \end{pmatrix} \tag{27}$$

Equations (18), (19), and (15a)–(15c) lead to

$$D_t(\beta) = \delta_{t0}[\alpha(\beta)]^{-1} + \delta_{t1}\phi(\beta)[\alpha(\beta)]^{-1} \tag{28}$$

$$G_t(\beta) = -\{\delta_{t0}[\alpha(\beta)]^{-1}\phi(\beta^{-1}) + \delta_{t1}[\alpha(\beta)]^{-1}\} \tag{29}$$

These two expressions are appropriately substituted for $X_t(\nu)$ and $Y_m(\beta)$ in Eqs. (24a) and (24b). The four elements of the matrix on the right side of Eq. (25) are thus expressed in terms of double contour integrals. Here (as well as in the evaluation of det λ'') the first integration is easily carried out in each matrix element. In the general λ'', the second integration leads to a function of α_1 and α_2 which is a mixture of algebraic terms and elliptic integrals. The result for λ' is much simpler. The terms with elliptic integrals completely cancel one another; the diagonal elements of the matrix on the right side of Eq. (25) vanish, while the off-diagonal elements (negatives of each other) are algebraic. The final result for the correlation function is

$$|\langle\sigma_{00}\sigma_{01}\sigma_{10}\rangle| = \mathcal{M}|\coth 2\beta E_1 + \coth 2\beta E_2 - \coth 2\beta E_1 \coth 2\beta E_2| \tag{30}$$

For $E_1 = E_2$, the right side of Eq. (30) reduces to Pink's result[4] for $|\langle\sigma_{00}\sigma_{01}\sigma_{02}\rangle|$. A plot of this common correlation function is given in Fig. 1.

The matrix elements on the right side of Eq. (26) may be simplified considerably by use of

$$(B_{2\alpha}S^{-1}C_{2\nu})_{pm} = (\gamma_{2\alpha,2\nu})_{p,m} - S^{-1}_{i_{2J}-i_{2\alpha}-1-p,\, i_{2J}-i_{2\nu}-m-1} \tag{31}$$

whose proof is outlined in the Appendix. Noting that the left side of Eq. (31)

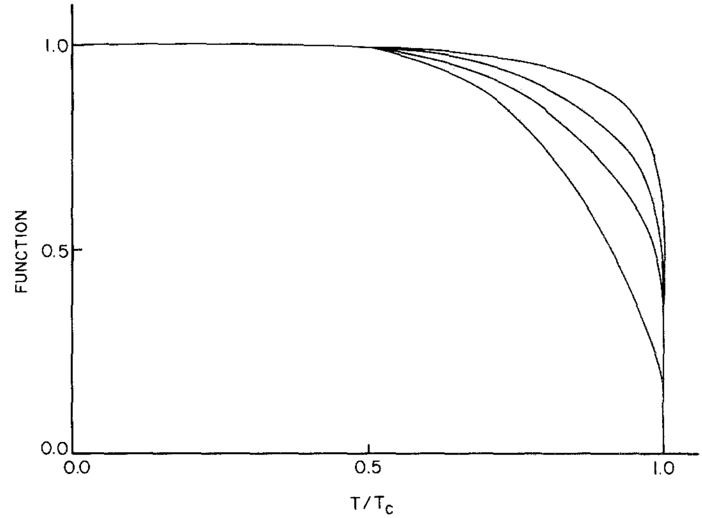

Fig. 1. Graphs (top to bottom) of $\mathscr{M}$, $|\langle\sigma_{00}\sigma_{01}\sigma_{02}\rangle|$, $|\sigma_{00}\sigma_{01}\sigma_{02}\sigma_{03}\sigma_{04}\rangle|$, and $|\langle\sigma_{00}\sigma_{02}\sigma_{04}\rangle|$ (all for $E_1 = E_2$) as functions of T. The second graph from the top also represents $|\langle\sigma_{00}\sigma_{01}\sigma_{10}\rangle|$ for $E_1 = E_2$.

is an element of $BS^{-1}C$ in Eq. (26) and that the corresponding element of γ is $(\gamma_{2\alpha,2\beta})_{pj}$, one sees that Eqs. (20a) and (26) reduce to

$$|\langle\sigma_{0i_0}\sigma_{0i_1}\cdots\sigma_{0i_{2J}}\rangle| = \mathscr{M}\det F \tag{32}$$

Each submatrix $F_{2\alpha,2\beta}$ of F has the dimensions of $\gamma_{2\alpha,2\beta}$, with matrix elements

$$(F_{2\alpha,2\beta})_{p,x} = S^{-1}_{i_{2J}-i_{2\alpha}-p-1,i_{2J}-i_{2\beta}-x-1} \tag{33}$$

Pink[4] found that $|\langle\sigma_{00}\sigma_{0j}\sigma_{0,j+1}\rangle| = \mathscr{M}(S^{-1})_{jj}$. For this case, his method is simpler than ours, but it is not readily extended to the more general cases discussed here, even for spins on a straight line. The present method immediately yields Pink's result for the equivalent function $|\langle\sigma_{00}\sigma_{01}\sigma_{0,j+1}\rangle|$; in this case the matrix F is one-dimensional. (As a check on the method, this result can be shown to be equal to the determinant of the j-dimensional F matrix appearing in $|\langle\sigma_{00}\sigma_{0j}\sigma_{0,j+1}\rangle|$.)

Figure 1 contains graphs of $|\langle\sigma_{00}\sigma_{01}\sigma_{02}\rangle|$, $|\langle\sigma_{00}\sigma_{01}\sigma_{02}\sigma_{03}\sigma_{04}\rangle|$, $|\langle\sigma_{00}\sigma_{02}\sigma_{04}\rangle|$, and $\mathscr{M}$, all for $E_1 = E_2$. As noted earlier, when $E_1 = E_2$, the first function equals $|\langle\sigma_{00}\sigma_{01}\sigma_{10}\rangle|$.

APPENDIX. DERIVATION OF EQ. (31)

From Eqs. (21c), (14b), and (14c) and the definition of $[B_{2\alpha}(v)]_l$,

$$(B_{2\alpha})_{l,q} = (2\pi i)^{-1} \oint [B_{2\alpha}(v)]_l v^{q-1}\, dv \tag{A1}$$

it follows that

$$[B_{2\alpha}(v)]_l = \phi(v) v^{(i_{2J} - i_{2\alpha} - l)} \tag{A2}$$

Similarly, if

$$(C_{2\beta})_{p,m} = (2\pi i)^{-1} \oint [C_{2\beta}(\lambda)]_m \lambda^{p-1}\, d\lambda \tag{A3}$$

then

$$[C_{2\beta}(\lambda)]_m = \phi(\lambda^{-1}) \lambda^{(i_{2J} - i_{2\beta} - m)} = [\phi(\lambda)]^{-1} \lambda^{(i_{2J} - i_{2\beta} - m)} \tag{A3}$$

From Eqs. (14c) and (24b) with $X_l(v) = [B_{2\alpha}(v)]_l$ and $Y_m(\lambda) = [C_{2\beta}(\lambda)]_m$, it follows that

$$\begin{aligned}(B_{2\alpha} S^{-1} C_{2\beta})_{l,m} \\ &= (2\pi i)^{-2} \oint\oint_{|\lambda v| < 1} \{(1 - \alpha_2 v^{-1})(1 - \alpha_1 \lambda^{-1})[(1 - \alpha_1 v^{-1})(1 - \alpha_2 \lambda^{-1})]^{-1}\}^{1/2} \\ &\quad \times [\lambda v(1 - \lambda v)]^{-1} v^{(i_{2J} - i_{2\alpha} - l)} \lambda^{(i_{2J} - i_{2\beta} - m)}\, d\lambda\, dv\end{aligned} \tag{A5}$$

After making the change of variable $\lambda' = \lambda^{-1}$, we find the λ' integration has two contributions. The first, arising from the pole at $\lambda' = v$, is equal to $(\gamma_{2\alpha,2\beta})_{lm}$ and exactly cancels the corresponding matrix element of γ in Eq. (26). The second contribution comes from the residue at the origin and is equal to the second term on the right of Eq. (31).

A similar development leads to cancellation of all elliptic integral terms implicitly appearing on the right side of Eq. (25) for the case $\lambda'' = \lambda'$.

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